meability makes it possible to minimize the forward-propagating energy within the ferrite which is a necessary condition for minimum forward attenuation both in the ferrite and in the resistance card.

Conclusion

It has been shown that the dominant-mode RF electric field configurations for both directions of propagation may be used to choose the design parameters for the resistance-sheet isolator. It has been concluded that a thick ferrite slab nearly against the side wall operated in a region of negative effective permeability will give the best performance. These are the same conditions used in the laboratory[5] with the possible exception of the condition on effective permeability.

The condition $\gamma 4\pi M \approx \omega$ (from Fig. 6) requires that the ferrite magnetization be known fairly accurately at the temperature and dc field intensity that are to be used. Then a value of internal dc field (corrected for demagnetization) may be chosen for operation in the region of negative effective permeability.

The uncertainties in the values of $4\pi M$ and H_0 will require the experimental adjustment of the applied field in order 1) to operate far enough below resonance, 2) to minimize forward loss over the band, 3) to find a conveniently small applied field, and 4) to nearly saturate the ferrite.

Appendix

The transcendental expression of (3) is somewhat complicated and must be solved numerically. The function $P(\pm\beta)$ from Lax, Button, and Roth[1] is

$$P(\pm\beta) = -\frac{pr \pm q(p^2 + q^2 - r^2)^{1/2}}{p^2 + q^2} \tag{5}$$

where

$$p = \frac{1}{2}\left(\frac{k_a{}^2\mu_{\text{eff}}{}^2}{\mu_0{}^2} + \frac{\beta^2}{\theta^2} - k_m{}^2\right)$$

$$q = j\frac{\beta k_a \mu_{\text{eff}}}{\theta\mu_0}$$

$$r = \frac{1}{2}\left(\frac{k_a{}^2\mu_{\text{eff}}{}^2}{\mu_0{}^2} - \frac{\beta^2}{\theta^2} + k_m{}^2\right)\cos k_a(L-\delta) + k_m \cot(k_m\delta)\frac{k_a\mu_{\text{eff}}}{\mu_0}\sin k_a(L-\delta).$$

Acknowledgment

The author is greatly indebted to Dr. Benjamin Lax for several helpful discussions and for his suggestion of essential points that have been incorporated in this paper. He also wishes to thank Dr. Gerald S. Heller for his criticism of the theory, Richard N. Brown for his assistance with the exploratory computations, and Mrs. Billie H. Houghton for computation of the final data.

Reciprocity Relationships for Gyrotropic Media*

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***Summary*—Reversal of the dc magnetic field in gyrotropic media transposes the tensor permeability and permittivity. It is shown that this also transposes the impedance, admittance, and scattering matrices of any device. It follows from this that the usual reciprocity statements for isotropic media apply to gyrotropic media if one reverses the dc magnetic field whenever an interchange of source and measurer is made.**

Introduction

FERRITES and gaseous plasma have been called "nonreciprocal" media because the usual reciprocity theorem[1] does not apply to them. However, a modified reciprocity theorem, stated by Rumsey and attributed to M. H. Cohen,[2] applies to such media. A number of useful and interesting interpretations of this reciprocity theorem are presented in this paper.

A ferrite in a dc magnetic field is characterized, insofar as an ac field is concerned, by a tensor permeability $[\mu]$ and a scalar permittivity ϵ.[3] Both $[\mu]$ and ϵ are independent of the amplitude of the ac field so long as it is sufficiently small. The $[\mu]$ is transposed if the dc magnetic field is reversed. A gaseous plasma in a dc magnetic field is characterized, insofar as an ac field is concerned, by a tensor permittivity $[\epsilon]$ and a scalar permeability μ.[4] Both $[\epsilon]$ and μ are independent of the amplitude of the ac field so long as it is sufficiently small. The $[\epsilon]$ is transposed if the dc magnetic field is reversed. The term *gyrotropic* is used to denote a medium characterized by $[\epsilon]$ and $[\mu]$, independent of the amplitude of an

* Manuscript received by the PGMTT, December 23, 1957; revised manuscript received, January 29, 1958.

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[1] S. A. Schelkunoff, "Electromagnetic Waves," McGraw-Hill Book Co., Inc., New York, N. Y., p. 478; 1943.

[2] V. H. Rumsey, "The reaction concept in electromagnetic theory," *Phys. Rev.*, vol. 94, pp. 1483–1491; June 15, 1954. Errata, vol. 95, p. 1705; September 15, 1954.

[3] D. Polder, "On the theory of ferromagnetic resonance," *Phil. Mag.*, vol. 40, pp. 99–115; January, 1949.

[4] H. Suhl and L. R. Walker, "Topics in guided wave propagation through gyrotropic media, part I," *Bell Sys. Tech. J.*, vol. 33, pp. 579–659; May, 1954.

ac field, and having the property that $[\epsilon]$ and $[\mu]$ are transposed by reversing the dc magnetic field. Ferrites and plasma are special cases of gyrotropic media. In subsequent discussion, the terminology "when $[\epsilon]$ and $[\mu]$ are transposed" will be frequently used. In ferrites and plasma, this is equivalent to saying "when the dc magnetic field is reversed."

The Reciprocity Theorem

The equations for an ac field in gyrotropic media are

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -j\omega[\mu]\boldsymbol{H} \\ \nabla \times \boldsymbol{H} &= j\omega[\epsilon]\boldsymbol{E} + \boldsymbol{J} \end{aligned} \tag{1}$$

where $\boldsymbol{J}$ represents the electric sources. All dissipation is accounted for in the $[\epsilon]$ and $[\mu]$, including conduction loss. The field from the same sources when $[\epsilon]$ and $[\mu]$ are transposed (indicated by $\sim$) will be denoted by $\widehat{\boldsymbol{E}}$, $\widehat{\boldsymbol{H}}$. Thus, the equations for the careted fields are

$$\begin{aligned} \nabla \times \widehat{\boldsymbol{E}} &= -j\omega[\tilde{\mu}]\widehat{\boldsymbol{H}} \\ \nabla \times \widehat{\boldsymbol{H}} &= j\omega[\tilde{\epsilon}]\widehat{\boldsymbol{E}} + \boldsymbol{J}. \end{aligned} \tag{2}$$

Now consider two sets of sources $\boldsymbol{J}_a$ and $\boldsymbol{J}_b$, producing fields $\boldsymbol{E}_a$, $\boldsymbol{H}_a$ and $\boldsymbol{E}_b$, $\boldsymbol{H}_b$ in the original medium, and fields $\widehat{\boldsymbol{E}}_a$, $\widehat{\boldsymbol{H}}_a$ and $\widehat{\boldsymbol{E}}_b$, $\widehat{\boldsymbol{H}}_b$ in the medium when $[\epsilon]$ and $[\mu]$ are transposed. By scalarly multiplying the second equation of (2) for the a field by $\boldsymbol{E}_b$, and the first equation of (1) for the b field by $\widehat{\boldsymbol{H}}_a$, one obtains

$$\begin{aligned} \boldsymbol{E}_b \cdot \nabla \times \widehat{\boldsymbol{H}}_a &= j\omega \boldsymbol{E}_b \cdot [\tilde{\epsilon}]\widehat{\boldsymbol{E}}_a + \boldsymbol{J}_a \cdot \boldsymbol{E}_b \\ \widehat{\boldsymbol{H}}_a \cdot \nabla \times \boldsymbol{E}_b &= -j\omega \widehat{\boldsymbol{H}}_a \cdot [\mu]\boldsymbol{H}_b. \end{aligned} \tag{3}$$

Subtract the second from the first and apply the identity $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$. This gives

$$\nabla \cdot (\widehat{\boldsymbol{H}}_a \times \boldsymbol{E}_b) = j\omega \boldsymbol{E}_b \cdot [\tilde{\epsilon}]\widehat{\boldsymbol{E}}_a + j\omega \widehat{\boldsymbol{H}}_a \cdot [\mu]\boldsymbol{H}_b + \boldsymbol{J}_a \cdot \boldsymbol{E}_b. \tag{4}$$

A similar manipulation of the second equation of (1) and the first of (2) yields

$$\nabla \cdot (\boldsymbol{H}_b \times \widehat{\boldsymbol{E}}_a) = j\omega \widehat{\boldsymbol{E}}_a \cdot [\epsilon]\boldsymbol{E}_b + j\omega \boldsymbol{H}_b \cdot [\tilde{\mu}]\widehat{\boldsymbol{H}}_a + \boldsymbol{J}_b \cdot \widehat{\boldsymbol{E}}_a. \tag{5}$$

In view of the identity $\boldsymbol{A} \cdot [a]\boldsymbol{B} = \boldsymbol{B} \cdot [\tilde{a}]\boldsymbol{A}$, subtraction of (5) from (4) gives

$$\nabla \cdot (\widehat{\boldsymbol{H}}_a \times \boldsymbol{E}_b - \boldsymbol{H}_b \times \widehat{\boldsymbol{E}}_a) = \boldsymbol{J}_a \cdot \boldsymbol{E}_b - \boldsymbol{J}_b \cdot \widehat{\boldsymbol{E}}_a. \tag{6}$$

Now integrate this throughout a region and apply the divergence theorem to the left-hand side. One now has

$$\iint (\widehat{\boldsymbol{H}}_a \times \boldsymbol{E}_b - \boldsymbol{H}_b \times \widehat{\boldsymbol{E}}_a) \cdot ds = \iiint (\boldsymbol{J}_a \cdot \boldsymbol{E}_b - \boldsymbol{J}_b \cdot \widehat{\boldsymbol{E}}_a) dv \tag{7}$$

where the surface is that bounding the region. Eq. (7) is the general reciprocity formula when all sources are of the electric type. This formula can be extended to include magnetic sources,[2] but these are not necessary for the objectives of this paper.

The left-hand side of (7) vanishes when the surface bounding the region recedes to infinity, or when the surface is covered by a perfect electric conductor. (It also vanishes under other conditions, but these two have the greatest physical significance.) The reciprocity theorem then reduces to

$$\iiint \boldsymbol{J}_a \cdot \boldsymbol{E}_b dv = \iiint \boldsymbol{J}_b \cdot \widehat{\boldsymbol{E}}_a dv. \tag{8}$$

Note that this reduces to the usual reciprocity theorem[1] when $[\epsilon]$ and $[\mu]$ are symmetric tensors, for then $\widehat{\boldsymbol{E}} = \boldsymbol{E}$. Rumsey has defined the quantities appearing in (8) to be *reactions*,[2] and interpreted them as "physical observables." Note that they are *not* power, which would require integrands of the form $\boldsymbol{J}^* \cdot \boldsymbol{E}$ (the * denotes complex conjugate). By defining the reactions

$$\begin{aligned} \langle a, b \rangle &= \iiint \boldsymbol{J}_a \cdot \boldsymbol{E}_b dv \\ \langle b, \hat{a} \rangle &= \iiint \boldsymbol{J}_b \cdot \widehat{\boldsymbol{E}}_a dv \end{aligned} \tag{9}$$

one can write the reciprocity theorem as

$$\langle a, b \rangle = \langle b, \hat{a} \rangle. \tag{10}$$

Thus, in the terminology of the reaction concept, the reciprocity theorem is: *The reaction of one set of sources on another is equal to the reaction of the latter set on the former when $[\epsilon]$ and $[\mu]$ are transposed.* The interpretation of this should become clearer when we apply it to networks in the next section.

Network Reciprocity

The current source of circuit theory is equivalent to a short filament of electric current applied between a pair of terminals. For this, the reaction becomes

$$\langle a, b \rangle = I_a \int \boldsymbol{E}_b \cdot d\boldsymbol{l}_a = I_a V_{ab} \tag{11}$$

where I_a is the a source and V_{ab} is the voltage due to the b source appearing across the a terminals. Note that this reaction is proportional to V_{ab} (equal to it if $I_a = 1$), and is therefore a measure of V_{ab}. Now consider an N terminal-pair network. The term "network" is used in its general sense, meaning "a configuration of matter." For linear matter, the properties of the network can be expressed in terms of an impedance matrix $[z]$ according to[5]

$$[V] = [z][I] \tag{12}$$

where $[V]$ and $[I]$ are the column matrices of the terminal voltages and currents. If a current I_i is applied to the ith terminal pair, and a current I_j to the jth terminal pair, we have $\langle i, j \rangle = I_i V_{ij}$ from (11). But the source I_j sees all terminals open circuited, so from (12) it follows that $V_{ij} = z_{ij} I_j$, where z_{ij} is the element of $[z]$ in the ith row, jth column. Thus, the elements of the impedance matrix in terms of the reactions are

[5] C. G. Montgomery, R. H. Dicke, E. M. Purcell, "Principles of Microwave Circuits," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 8, p. 87; 1948.

$$z_{ij} = \frac{\langle i, j \rangle}{I_i I_j} \tag{13}$$

valid for all i and j (including $i=j$). The reactions between unit currents are numerically equal to the impedance elements.

Now let the network contain gyrotropic media. Define the impedance matrix for the network with $[\epsilon]$ and $[\mu]$ transposed according to

$$[\hat{V}] = [\hat{z}][I] \tag{14}$$

where $[\hat{V}]$ is the column matrix of the new terminal voltages. Following the same reasoning as for the original network, one has

$$\hat{z}_{ij} = \frac{\langle i, \hat{j} \rangle}{I_i I_j}\,. \tag{15}$$

From (10), (13), and (15), it is evident that $z_{ij} = \hat{z}_{ji}$, or

$$[\hat{z}] = [\tilde{z}]. \tag{16}$$

Thus, the network reciprocity theorem is: *The impedance matrix of any network is transposed when* $[\epsilon]$ *and* $[\mu]$ *are transposed.*

Corollaries to this are: *The admittance matrix and the scattering matrix of any network are transposed when* $[\epsilon]$ *and* $[\mu]$ *are transposed.* It is evident that the admittance matrix $[y]$, defined according to[5]

$$[I] = [y][V], \tag{17}$$

is transposed, since $[y]$ is the inverse matrix of $[z]$. The transposition of $[y]$ can also be shown by a procedure analogous to that used to show the transposition of $[z]$. For this, one should extend the reciprocity theorem to magnetic sources, and use the magnetic current equivalent of the circuit voltage source.[6] The scattering matrix $[S]$ is defined according to[7]

$$[V^s] = [S][V^i] \tag{18}$$

where $[V^i]$ is the column matrix of the incident components of terminal voltage and $[V^s]$ is the column matrix of the reflected components. In terms of the impedance matrix, one has[7]

$$[S] = [z - I][z + I]^{-1} \tag{19}$$

where $[I]$ is the unit matrix. The proof that $[S]$ is transposed when $[z]$ is transposed is outlined as follows.

$$\begin{aligned}[\hat{S}] &= [\tilde{z} - I][\tilde{z} + I]^{-1} = [\widetilde{z - I}][\widetilde{z + I}]^{-1} \\ &= \widetilde{[z + I]^{-1}[z - I]} = \widetilde{[z - I][z + I]^{-1}} = [\tilde{S}].\end{aligned} \tag{20}$$

The theorem on the transposition of network matrices applies to microwave networks as well as to the usual circuit theory networks. For example, if the input terminals (or ports) to the network are waveguides, one replaces the current elements used in the previous proof by current sheets across the waveguides. These current sheets are chosen so that only the desired modes are excited. The remainder of the proof is then identical to the preceding one.

Interpretations of Reciprocity

A few specific interpretations will now be given so that the reader may appreciate the significance of the above reciprocity theorems. These will be stated in terms of reversing the dc magnetic field in gyrotropic media to emphasize the physical interpretation of transposing $[\epsilon]$ and $[\mu]$. Perhaps it should be pointed out that the dc field need not be uniform. Reversal of the dc magnetic field applies to all points, being roughly equivalent to reversing the dc current in a field winding.

1) *The input impedance to any device containing gyrotropic media is unchanged by reversing the dc magnetic field.* The impedance matrix of a two-terminal (one-port) device is of order one (a single element), and thus equal to its transpose.

2) *For a network containing gyrotropic media, the positions of an ac current source and an infinite impedance voltmeter may be interchanged without affecting the voltmeter reading, if at the same time the dc magnetic field is reversed.* This follows from the transposition of $[z]$, since the voltage at terminals i due to a current in terminals j is $V_i = z_{ij} I_j$, all terminals open-circuited.

3) *For a network containing gyrotropic media, the positions of an ac voltage source and an impedanceless ammeter may be interchanged without affecting the ammeter reading, if at the same time the dc magnetic field is reversed.* This follows from the transposition of $[y]$, since the current at terminals i due to a voltage across terminals j is $I_i = y_{ij} V_j$, all terminals short-circuited.

4) *The resonant frequencies of any cavity containing gyrotropic media are unchanged by reversing the dc magnetic field.* This follows from statement 1), since the resonances are infinities of the input impedance.

5) *The transmitting pattern of any antenna containing gyrotropic media is the same as the receiving pattern with the dc magnetic field reversed.* This follows from statement 2) if one considers the given antenna plus a "test" antenna as forming a two-terminal pair device.

6) *The cut-off frequency spectrum of any cylindrical waveguide containing gyrotropic media is unchanged when the dc magnetic field is reversed.* This follows from statement 4), since the cut-off frequencies are the resonances of a two-dimensional cavity.

7) *In any cylindrical waveguide containing gyrotropic media, the propagation constants of positive traveling modes are interchanged with those of negative traveling modes when the dc magnetic field is reversed.* The proof of this follows from statement 2) if one considers the two ends of the guide as ports to a microwave network.

Acknowledgment

This work was supported in part by Office of Ordinance Research, U. S. Army, Contract No. DA-30-115-ORD-861.

[6] R. F. Harrington, "Field equivalence theorems and their circuit analogues," *Elec. Eng.*, vol. 73, pp. 923–927; October, 1954.

[7] Montgomery, Dicke, and Purcell, *op. cit.*, p. 146.